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Vortex Motion in certain Triangles.

By A. E. H. Love, B. A., *Fellow of St. John's College, Cambridge.*

1. In the Proceedings of the London Mathematical Society, Vol. XII, Routh has explained a method whereby the motion of a fluid in two dimensions due to sources or vortices situated within a given region can be inferred from the motion due to sources or vortices of equal strength placed at the corresponding points of another region, whenever, by means of a transformation by conjugate functions, we can obtain a correlation* ("conforme Abbildung") of the first region with the second in such a way that elementary portions of the two regions are similar. The conditions for such correlation may be expressed as follows: Let Z be a function of a complex variable z, and suppose Z and z represented in the usual manner by points on two planes, then the boundary of any region in the z plane will be transformed into a certain curve in the Z plane. The function Z is to be chosen to be finite, continuous, and one-valued within the region in the z plane, and points on the boundary in the z plane are to correspond to points on the boundary in the Z plane in such a way that *one* point on either boundary corresponds to *one* point on the other.

If Φ and Ψ are the velocity- and stream-functions of any fluid motion in the Z plane, then $\Phi + \iota\Psi$ is a function of Z, and Routh shows that the velocity- and stream-functions in the corresponding motion within the correlated region of the Z plane can be inferred by simply substituting for Z in $\Phi + \iota\Psi$ its value expressed as a function of z.

In Art. 21 of Riemann's "Inaugural Dissertation" (Ges. Werke, p. 40) it is stated that the problem to find a correlation of two simply-connected plane regions has always one and only one solution,* and thus in particular it is always theoretically possible to correlate any given simply-connected plane region with that part of a plane which lies on one side of the axis of real quantities. In a memoir in Borchardt's Journal (Bd. 70), Schwarz has discussed the solution of the problem when the region to be correlated with the half plane is bounded by a polygon, and he has shown that when the polygon is a triangle in the z plane, the function Z is in general a transcendent having an infinite number of values at every point of the z plane, of which the value along any given branch of the function is, however, singly determinate (eindeutig) within the triangle. But there are certain exceptional cases in which Z is an algebraic function of an elliptic function $\wp z$ of z, $\wp z$ being one-valued at every point of the z plane. These exceptional cases are those of a triangle whose angles are

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* For the theory of the correlation of two plane regions the reader may consult Darboux, "Théorie Générale des Surfaces," ch. IV, or Holzmüller, "Einführung in die Theorie der isogonalen Verwandtschaften und der conformen Abbildungen."

$$\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right), \left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}\right), \left(\frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{6}\right), \text{ or } \left(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}\right).$$

The integration of the differential equation on which the correlation depends had been considered by Briot and Bouquet in their "Théorie des Fonctions doublement périodiques."†

In what follows I propose to work out in detail these cases, and to apply the solution to the problem of the motion of a single vortex within the triangle. The solution for any number of vortices can be found by summation, while that for sources is obtained by interchanging Φ and Ψ.

2. Schwarz shows that if a, b, c are points on the real axis in the Z plane, then a triangle whose angles are $\alpha\pi$, $\beta\pi$, $\gamma\pi$ in the z plane can be correlated with the half plane above the real axis in the Z plane by means of the equation

$$\frac{d}{dZ}\left(\log\frac{dz}{dZ}\right) = \frac{\alpha-1}{Z-a} + \frac{\beta-1}{Z-b} + \frac{\gamma-1}{Z-c},$$

and the corners of the triangle will correspond to the points $Z = a, b, c$.

This appears because (1) the function z of Z is plainly in any correlation of the form $C_1 z + C_2$, where C_1, C_2 are arbitrary constants, and z is one determination of the function; (2) the transformation

$$\frac{d}{dZ}\left(\log\frac{dz}{dZ}\right) = \frac{\alpha-1}{Z-a}$$

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*All solutions are counted as one solution in which *finite* triangles remain similar.

† See the same author's "Théorie des Fonctions Elliptiques," pp. 388 seq.

effects a correlation of the half plane Z (imaginary part positive) with the space bounded by two lines in the z plane inclined to each other at angle $\alpha\pi$, and meeting in the point corresponding to $Z=a$.

It follows that the desired correlation is effected by means of the equation

$$C_1z+C_2=\int^z(Z-a)^{\alpha-1}(Z-b)^{\beta-1}(Z-c)^{\gamma-1}dZ, \tag{A}$$

in which the points a, b, c are taken to lie in a definite order on the real axis, viz. if A, B, C be the order of the corners of the triangle in the z plane at which the angles are $\alpha\pi$, $\beta\pi$, $\gamma\pi$ when its contour is described in the positive sense, *i. e.* with its area on the left, then a, b, c is to be the order of the points in the Z plane which correspond to the corners, the real axis being described in the positive sense.

CASE I.—*The equilateral triangle.*

3. In this case the equation (A) becomes

$$C_1z+C_2=\int^z\frac{dZ}{(Z-a)^{\frac{2}{3}}(Z-b)^{\frac{2}{3}}(Z-c)^{\frac{2}{3}}}. \tag{1}$$

We may take $a=0$, $b=1$, $c=-1$; then, supposing $Z=-\dfrac{1}{Z'}$, we have

$$C_1z+C_2=\int^{z'}\frac{dZ'}{(Z'^2-1)^{\frac{2}{3}}}.$$

Write

$$1-Z'^2=x^3=1-\frac{1}{Z^2}, \tag{2}$$

thus

$$C_1z+C_2=-\frac{3}{2}\int^x\frac{dx}{\sqrt{1-x^3}},$$

or

$$-\frac{1}{3}(C_1z+C_2)=\int^x\frac{dx}{\sqrt{4(x^3-1)}}.$$

Hence, adjusting constants, we may write

$$x=\wp z, \tag{3}$$

where $\wp z$ is defined by the equation

$$\left.\begin{aligned}\wp'^2z&=4(\wp^3z-1)=4(\wp z-e_1)(\wp z-e_2)(\wp z-e_3),\\ e_1&=-\frac{1}{2}+\iota\frac{\sqrt{3}}{2},\ e_2=1,\ e_3=-\frac{1}{2}-\iota\frac{\sqrt{3}}{2}.\end{aligned}\right\} \tag{4}$$

The correlation is effected by means of the equation

$$1-\frac{1}{Z^2}=\wp^3z,$$

giving
$$Z^2=\frac{1}{1-\wp^3z}=\frac{-1}{(\wp z-e_1)(\wp z-e_2)(\wp z-e_3)},$$

thus
$$-\frac{\iota Z}{2}=\frac{1}{\wp' z}. \tag{5}$$

4. The elliptic functions given by (4) have been discussed by Greenhill.* We have to notice that the discriminant is negative and equal to -432. The real half-period ω_2 is given by the equation

$$\omega_2=\int_1^\infty\frac{dx}{\sqrt{4(x^3-1)}}=\frac{1}{6}\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2}{3}\right)},$$

the corresponding imaginary half-period $\omega_2'=\iota\omega_2\sqrt{3}$. The periods ω_1, ω_3 corresponding to e_1, e_3 are $\omega_1=\frac{1}{2}(\omega_2-\omega_2')$, $\omega_3=\frac{1}{2}(\omega_2+\omega_2')$.

We can also express in a simple form all the roots of the function $\wp z$. We know that for pure imaginary values of z, $\wp z$ passes through all real values between $e_2(=1)$ and $-\infty$, and takes the same value at points whose z differ by multiples of $2\omega_2'$. Hence there is a root for some value of z a pure imaginary between 0 and $2\omega_2'$. Now we may show that if $\wp z=0$, then $\wp(3z)=\infty$, this follows at once from the formula

$$\wp 3z-\wp z=-\frac{\wp'^2z\,(2\wp^6z-40\wp^3z-16)}{3\,(\wp^4z-4\wp z)}$$

by putting $\wp z=0$. Thus it appears that $\frac{2}{3}\omega_2'$ is a root of $\wp z$ which is pure imaginary and lies between 0 and $2\omega_2'$, the other root $\frac{4}{3}\omega_2'$ may be regarded as $-\frac{2}{3}\omega_2'+2\omega_2'$, so that it differs from the former by a period and a change of sign.

5. Returning to equation (5), we see that $\wp^3z=0$ gives the values of z which correspond to $Z=\pm 1$, and $\wp z=\infty$ corresponds to $Z=0$. We may choose

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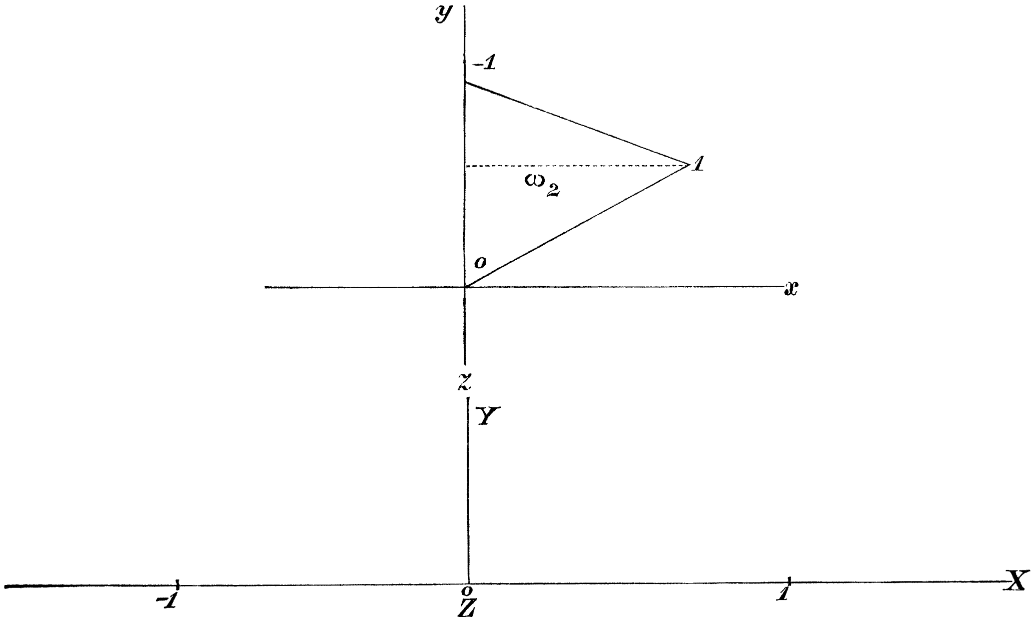
* "The Trajectory for the Cubic Law of Resistance," Proc. Royal Artillery Institution, 1886.

the pole z for $Z=0$ to be $z=0$, and the root z for $Z=-1$ to be $z=\frac{2}{3}\omega_2'$; then since $-\frac{2}{3}\omega_2'=\left(\omega_2+\frac{1}{3}\omega_2'\right)-2\omega_3$, we have $\wp\left(\omega_2+\frac{1}{3}\omega_2'\right)=0$, and $z=\omega_2+\frac{1}{3}\omega_2'$ may be taken to correspond to $Z=1$.

The function $Z=2\iota/\wp' z$ is finite, continuous, and one-valued within the equilateral triangle in the z plane whose corners have coordinates

$$(0,0),\ \left(0,\frac{2}{\sqrt{3}}\omega_2\right),\ \left(\omega_2,\frac{1}{\sqrt{3}}\omega_2\right),$$

and the points on the real axis in the Z plane correspond to points on the sides of the triangle, the points $Z=0,\ 1,\ -1$ being those which correspond to the corners. The height of the triangle is the real half-period ω_2.



The figure represents the boundaries of the two correlated regions, the points $0,\ 1,\ -1$ correspond to the angular points of the triangle and these are marked with the same numbers in both figures.

6. Now let z_0 be the point at which is a vortex of strength m, and let Z_0 be the corresponding point of the Z plane, the imaginary part of Z_0 being supposed

positive. Then if Z_0', z_0' be the conjugate imaginaries of Z_0, z_0, the fluid motion is given by

$$\Psi + \iota\Phi = \frac{m}{2\pi}\log\frac{Z - Z_0}{Z - Z_0'} = \frac{m}{2\pi}\log\frac{\wp' z - \wp' z_0}{\wp' z - \wp' z_0'} \tag{6}$$

The position of z_0 is given by $\wp^3 z_0 = 1 - \frac{1}{Z_0^2}$, and if z_0 be one of the roots of this equation, there will be other roots which do not differ from z_0 by multiples of the periods. There are in fact two such roots z_1, z_2 for which $\wp z_1 = e_1 \wp z_0$, $\wp z_2 = e_3 \wp z_0$, and then $z_1 = e_1 z_0$ and $z_2 = e_3 z_0$.

This appears from the equation

$$\left(\frac{d\,(e_1 \wp z)}{d\,(e_1 z)}\right)^2 = 4\,(\wp^3 z - 1) = 4\left\{(e_1 \wp z)^3 - 1\right\}$$

e_1 being a cube root of unity, from which we deduce

so
$$\left.\begin{aligned} \wp e_1 z &= e_1 \wp z \\ \wp e_3 z &= e_3 \wp z \end{aligned}\right\} \qquad (7)[*]$$

The roots z_0, z_1, z_2 of the equation $\wp^3 z = 1 - \frac{1}{Z_0^2}$ satisfy the relation $z_0 + z_1 + z_2 = 0$. If z_0' be the conjugate imaginary to z_0, then the conjugate imaginaries z_1', z_2' to z_1, z_2 are $e_3 z_0'$, $e_1 z_0'$.

Now consider the function

$$\phi_2(z) = \begin{vmatrix} 1 & \wp z & \wp' z \\ 1 & \wp z_1 & \wp' z_1 \\ 1 & \wp z_2 & \wp' z_2 \end{vmatrix} = (\wp z_2 - \wp z_1)(\wp' z - \wp' z_0), \tag{8}$$

where $\wp' z_1 = \wp' z_2 = \wp' z_0$.

We know that

$$\phi_2(z) = 2\,\frac{\sigma(z_1 - z)\,\sigma(z_2 - z)\,\sigma(z + z_1 + z_2)\,\sigma(z_2 - z_1)\dagger}{\sigma^3 z\,\sigma^3 z_1\,\sigma^3 z_2}\,. \tag{9}$$

Hence, remembering that σz is an odd function, and that

$$\wp z_2 - \wp z_1 = -\,\frac{\sigma(z_2 + z_1)\,\sigma(z_2 - z_1)}{\sigma^2 z_2\,\sigma^2 z_1}\,, \tag{10}$$

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[*] Greenhill, loc. cit. Art. 6.

[†] Halphen, "Traité des Fonctions Elliptiques," p. 219.

we obtain

$$\wp' z - \wp' z_0 = 2\frac{\sigma(z-z_0)\,\sigma(z-z_1)\,\sigma(z-z_2)}{\sigma^3 z\,\sigma z_0\,\sigma z_1\,\sigma z_2}, \tag{11}$$

so that the fluid motion is given by

$$\Psi + \iota\Phi = \frac{m}{2\pi}\log\frac{\sigma(z-z_0)\,\sigma(z-z_1)\,\sigma(z-z_2)}{\sigma(z-z_0')\,\sigma(z-z_1')\,\sigma(z-z_2')}. \tag{12}$$

7. Greenhill* has considered this case of fluid motion proceeding by the method of images. He observes that the images are

$$e_1 z_0,\ e_3 z_0,\ z_0',\ e_1 z_0',\ e_3 z_0'$$

and the similar sets grouped round centres of hexagons whose coordinates are $2m\omega_2$, $2m\omega_2\sqrt{3}$; $(2m+1)\,\omega_2$, $(2m+1)\,\omega_2\sqrt{3}$. As the vectors of all these are included in the formula $2m\omega_1 + 2m'\omega_3$, the solution found by summation will be that in (12), since

$$\sigma z = z\pi_w'\left(1-\frac{z}{w}\right)e^{\frac{z}{w}+\frac{1}{2}\frac{z^2}{w^2}},\dagger$$

where w has all the values taken by the quantity $2m\omega_1 + 2m'\omega_3$, when m, m' are integers varying from 0 to $\pm\infty$, except the value 0. In Greenhill's solution, which differs from (12) in form, the σ functions are formed with ω_2 and ω_2' as fundamental half-periods, and, since $\omega_2 + \omega_2'$ is a period, the σ_2 so formed is included in the σ formed with ω_1, ω_3 as fundamental half-periods.

CASE II.—*The right-angled triangle containing an angle of* 60°.

8. In this case equation (A) becomes

$$C_1 z + C_2 = \int^z \frac{dZ}{(Z-a)^{\frac{1}{2}}(Z-b)^{\frac{2}{3}}(Z-c)^{\frac{5}{6}}}. \tag{13}$$

Take $a=0$, $b=1$, $c=-1$, and suppose $z=\dfrac{1}{Z'}$, then

$$C_1 z + C_2 = -\int^{Z'} \frac{dZ'}{(1-Z')^{\frac{2}{3}}(1+Z')^{\frac{5}{6}}}.$$

Write

$$\frac{Z'-1}{Z'+1} = x^3 = \frac{1-Z}{1+Z}, \tag{14}$$

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* "Applications of Weierstrass's Elliptic Functions," Proc. Lond. Math. Soc., Vol. XVIII, p. 377.

† Schwarz, "Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen," p. 5.

then

$$C_1 z + C_2 = -\frac{3}{2}\int^x \sqrt{\frac{2}{1-x^3}}\,dx$$

$$= 3\sqrt{2}\iota\int^x \frac{dx}{4(x^3-1)}. \qquad (15)$$

Hence, adjusting constants,

$$x = \wp z, \qquad (16)$$

where the invariants and periods of the elliptic functions are the same as in Case I.

The correlation is effected by means of the equation

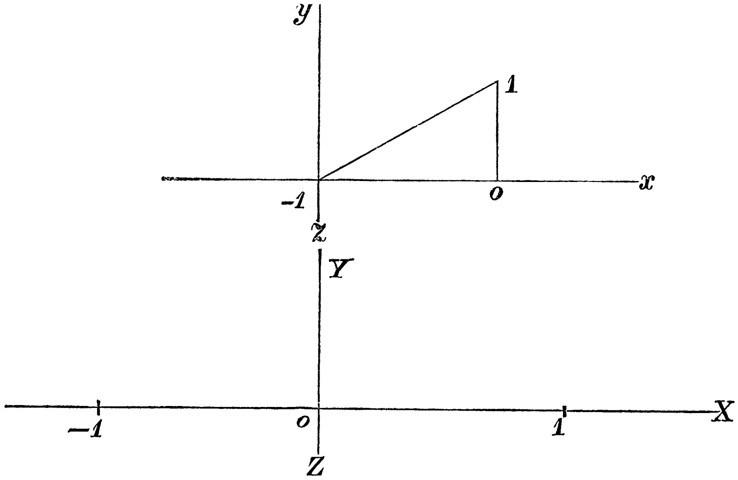
$$\frac{1-Z}{1+Z} = \wp^3 z,$$

or

$$Z = \frac{1-\wp^3 z}{1+\wp^3 z}. \qquad (17)$$

The point $Z=0$ corresponds to $\wp^3 z = 1$ or $\wp' z = 0$, giving $z = \omega_1$, ω_2, or ω_3, we may choose $z = \omega_2$. The point $Z = 1$ corresponds to $\wp^3 z = 0$, we may choose $z = \omega_2 + \frac{1}{3}\omega_2'$. The point $Z = -1$ corresponds to $\wp^3 z = \infty$, and we may choose $z = 0$.

The function $Z = (1-\wp^3 z)/(1+\wp^3 z)$ is finite, continuous, and one-valued within the right-angled triangle in the z plane whose corners have coordinates $(0, \omega_2)$, $\left(\omega_2, \frac{1}{\sqrt{3}}\omega_2\right)$, $(0, 0)$, and the points on the real axis in the Z plane correspond to the points on the sides of the triangle, the points $Z = 0, 1, -1$ being those which correspond to the corners. The side opposite to the angle of 60° is of length equal to the real half-period ω_2.



The figures are drawn as in Case I.

9. Suppose there is a vortex of strength m at z_0, and let Z_0 be the corresponding point in the Z plane, and z_0', Z_0' the conjugate imaginaries to z_0, Z_0, then the fluid motion is given by

$$\Psi + \iota\Phi = \frac{m}{2\pi} \log \frac{Z - Z_0}{Z - Z_0'} = \frac{m}{2\pi} \log \frac{\wp^3 z - \wp^3 z_0}{\wp^3 z - \wp^3 z_0'} \,. \tag{18}$$

The roots of $\wp^3 z = \wp^3 z_0$ are z_0, $e_1 z_0$, $e_3 z_0$, say z_0, z_1, z_2, and those of $\wp^3 z = \wp^3 z_0'$ are the conjugate imaginaries, say z_0', z_1', z_2', resolving the numerator and denominator of (18) into factors and remembering the fundamental equation (10), we find

$$\Psi + \iota\Phi = \frac{m}{2\pi} \log \frac{\sigma(z - z_0)\,\sigma(z - z_1)\,\sigma(z - z_2)\,\sigma(z + z_0)\,\sigma(z + z_1)\,\sigma(z + z_2)}{\sigma(z - z_0')\,\sigma(z - z_1')\,\sigma(z - z_2')\,\sigma(z + z_0')\,\sigma(z + z_1')\,\sigma(z + z_2')} \,. \tag{19}$$

CASE III.—*The isosceles triangle containing an angle of* 120°.

10. In this case the equation (A) becomes

$$C_1 z + C_2 = \int^z \frac{dZ}{(Z - a)^{\frac{1}{3}}(Z - b)^{\frac{5}{6}}(Z - c)^{\frac{5}{6}}} \,. \tag{20}$$

Take $a = 0$, $b = 1$, $c = -1$, and suppose $z = \dfrac{1}{Z'}$, then

$$C_1 z + C_2 = \int^{z'} - \frac{dZ'}{(1 - Z'^2)^{\frac{5}{6}}} \,.$$

Write

$$x^3 = \frac{1}{1 - Z'^2} = \frac{Z^2}{Z^2 - 1} \,, \tag{21}$$

then

$$C_1 z + C_2 = -3 \int^x \frac{dx}{\sqrt{4(x^3 - 1)}} \,. \tag{22}$$

Hence, adjusting constants,

$$x = \wp z \,, \tag{23}$$

where the invariants and periods of the elliptic functions are the same as in Case I.

The correlation is effected by means of the equation

$$Z^2 = \frac{\wp^3 z}{\wp^3 z - 1} \,,$$

or

$$Z = 2 \frac{\wp^{\frac{3}{2}} z}{\wp' z} \,. \tag{24}$$

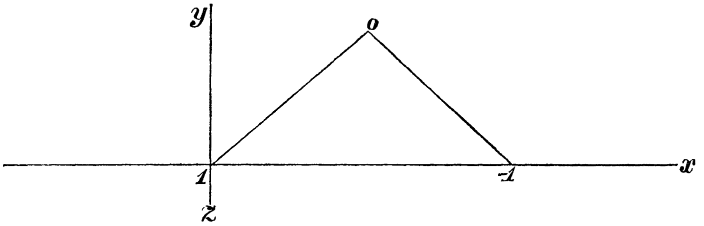
The point $Z = 0$ corresponds to $\wp z = 0$ and we may choose $z = \omega_2 + \frac{1}{3} \omega_2'$. The points $Z = \pm 1$ correspond to $\wp z = \infty$, and we may choose $z = 0$ for $Z = 1$

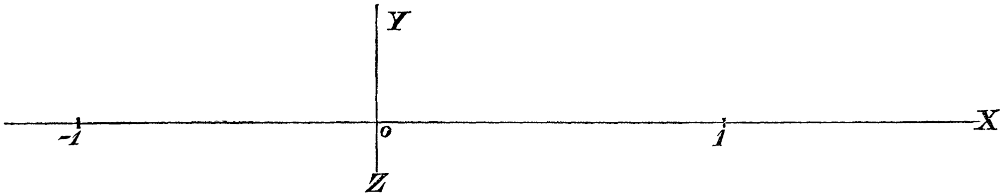
and $z = 2\omega_2$ for $Z = -1$. Then since the function $\wp z$ does not vanish anywhere within the triangle whose corners are $z = \omega_2 + \frac{1}{3}\omega_2'$, 0, $2\omega_2$, it follows that the function $\wp^{\frac{3}{2}}z$ has no branch-point within the triangle, and is consequently finite, continuous, and one-valued within the triangle when its value has been chosen for one point.

Thus, choosing one branch of the function $\wp^{\frac{3}{2}}z$, we may say that the function $Z = 2\wp^{\frac{3}{2}}z/\wp'z$ is finite, continuous, and one-valued within the isosceles triangle in the z plane whose corners have coordinates

$$\left(\omega_2, \frac{1}{\sqrt{3}}\omega_2\right), (0, 0), (2\omega_2, 0),$$

and the points on the real axis in the Z plane correspond to the points on the sides of the triangle, the points $Z = 0, 1, -1$ being those which correspond to the corners. The length of the base of the triangle is the real period $2\omega_2$.





The figures are drawn as in Case I.

11. Suppose there is a vortex of strength m at z_0, and let z_0' be the conjugate imaginary, Z_0, Z_0' the corresponding points in the Z plane. Then the fluid motion is given by

$$\Psi + \iota\Phi = \frac{m}{2\pi}\log\frac{Z - Z_0}{Z - Z_0'} = \frac{m}{2\pi}\log\left\{\left(\frac{\wp^{\frac{3}{2}}z}{\wp'z} - \frac{\wp^{\frac{3}{2}}z_0}{\wp'z_0}\right)\Big/\left(\frac{\wp^{\frac{3}{2}}z}{\wp'z} - \frac{\wp^{\frac{3}{2}}z_0'}{\wp'z_0'}\right)\right\} \quad (25)$$

The function $\frac{\wp^{\frac{3}{2}}z}{\wp'z} - \frac{\wp^{\frac{3}{2}}z_0}{\wp'z_0}$ is not a rational function of $\wp z$, $\wp'z$ and does not appear to be expressible as a product of σ functions.

Case IV.—*The isosceles right-angled triangle.*

12. In this case the equation (A) becomes

$$C_1z + C_2 = \int^z \frac{dZ}{(Z-a)^{\frac{1}{2}}(Z-b)^{\frac{3}{4}}(Z-c)^{\frac{3}{4}}}. \tag{26}$$

Take $a=0$, $b=1$, $c=-1$ and suppose $Z=\frac{1}{Z'}$, then

$$C_1z + C_2 = -\int^{z'} \frac{dZ'}{(1-Z'^2)^{\frac{3}{4}}}.$$

Write

$$\frac{1}{1-Z'^2} = x^2 = \frac{Z^2}{Z^2-1}, \tag{27}$$

then

$$C_1z + C_2 = -\int \frac{dx}{\sqrt{x(x^2-1)}}. \tag{28}$$

Hence, adjusting constants,

$$x = \wp z, \tag{29}$$

where $\wp z$ is defined by the equation

$$\left.\begin{aligned} &\wp'^2 z = 4(\wp z - e_1)(\wp z - e_2)(\wp z - e_3), \\ &e_1 = 1,\ e_2 = 0,\ e_3 = -1. \end{aligned}\right\} \tag{30}$$

The correlation is effected by means of the equation

$$\frac{Z^2}{Z^2-1} = \wp^2 z,$$

giving

$$Z = \frac{\wp z}{\sqrt{\wp^2 z - 1}} = \frac{\wp z - e_2}{\sqrt{(\wp z - e_1)(\wp z - e_3)}}. \tag{31}$$

15. In the elliptic functions given by (30) we have to notice that the discriminant is positive, and the invariant $\wp_3 = 0$. The real half-period ω is given by

$$\omega = \int_1^\infty \frac{dx}{\sqrt{4x(x^2-1)}} = \frac{1}{4}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)},$$

and the pure imaginary half-period $\omega' = \iota\omega$.

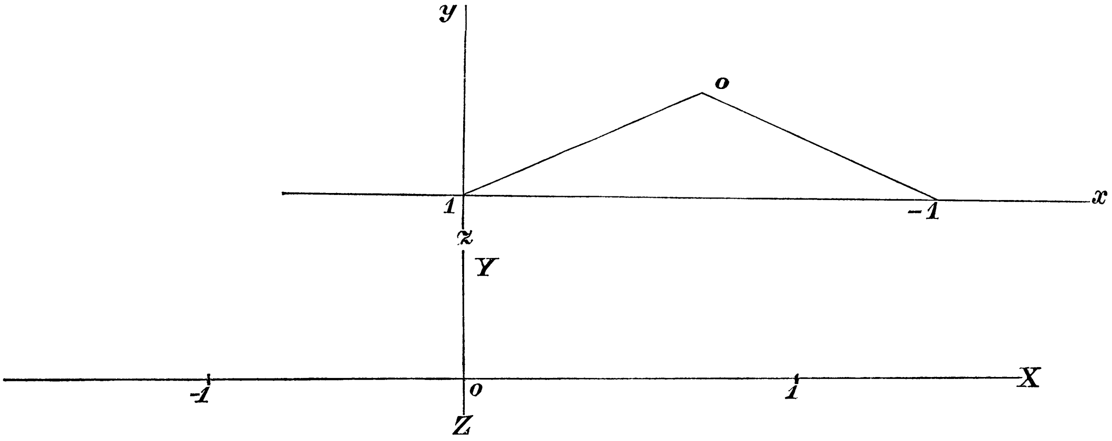
One root of $\wp z$ is easily found since $\wp(\omega + \omega') = e_2 = 0$.

Again $\sqrt{\wp z - e_1}$ and $\sqrt{\wp z - e_3}$ are singly determinate functions, viz. $\sqrt{\wp z - e_1} = \frac{\sigma_1 z}{\sigma z}$, and $\sqrt{\wp z - e_3} = \frac{\sigma_3 z}{\sigma z}$, and since $e_2 = 0$ we have in like manner $\wp z = \frac{\sigma_2^2 z}{\sigma^2 z}$. Hence

$$Z = \frac{\sigma_2^2 z}{\sigma_1 z \sigma_3 z}. \tag{32}$$

14. The point $Z = 0$ corresponds to $\wp z = 0$ and we may choose $z = \omega + \omega'$. The points $Z = \pm 1$ correspond to $\wp z = \infty$, and we may choose $z = 0$ for $Z = 1$, and $z = 2\omega$ for $Z = -1$.

The function $Z = \sigma_2^2 z/(\sigma_1 z \sigma_3 z)$ is finite, continuous, and one-valued within the isosceles right-angled triangle in the z plane whose corners have coordinates (ω, ω), $(0, 0)$, $(2\omega, 0)$, and points on the real axis in the Z plane correspond to points on the sides of the triangle, the points $Z = 0, 1, -1$ being those which correspond to the corners. The length of the hypothenuse is the real period 2ω.



The figures are drawn as in Case I.

15. Suppose there is a vortex of strength m at the point z_0, and let Z_0 be the corresponding point in the Z plane, Z_0' the image of Z_0 in the real axis and z_0' the corresponding point to Z_0', it will be necessary to prove that z_0, z_0' are conjugate imaginaries. We shall show that if z_0' be the conjugate imaginary to z_0, then Z_0' is conjugate to Z_0.

Since η* and ω are real, we see that the function $\sigma_1(\alpha + \iota\beta)$, being equal to $\frac{\sigma(\alpha + \iota\beta + \omega)}{\sigma\omega} e^{-\eta(\alpha + \iota\beta)}$, is the conjugate imaginary to $\sigma_1(\alpha - \iota\beta)$.

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*η is the constant $\frac{\sigma'\omega}{\sigma\omega}$ which $= \frac{1}{\omega} + \int_0^{\omega} \left(\frac{1}{x^2} - \wp x \right) dx$. The constant η' is $\frac{\sigma'\omega'}{\sigma\omega'}$, and the relations $\eta\omega' - \eta'\omega = \frac{1}{2}\iota\pi$, $\omega' = \iota\omega$, $\eta' = -\iota\eta$ have place (Halphen, ch. V).

The function $\sigma_2(\alpha+\iota\beta)=\dfrac{\sigma\{\alpha+\omega+\iota(\beta+\omega)\}}{\sigma(\omega+\iota\omega)}e^{-\eta(\alpha+\iota\beta)+\iota\eta(\alpha+\iota\beta)}$, and the exponential factor is $e^{-\eta(\alpha+\beta)-\iota\eta(\beta-\alpha)}$. Also

$$\begin{aligned}\sigma_2(\alpha-\iota\beta)&=\frac{\sigma\{(\alpha+\omega-\iota(\beta+\omega)+2\omega')\}}{\sigma(\omega-\iota\omega+2\omega')}e^{-\eta(\alpha-\beta)+\iota\eta(\beta+\alpha)}\\&=\frac{\sigma\{(\alpha+\omega)-\iota(\beta+\omega)\}}{\sigma(\omega-\iota\omega)}e^{2\eta'(\alpha-\iota\beta)+\iota\eta(\beta-\alpha)}\\&=\frac{\sigma\{(\alpha+\omega)-\iota(\beta+\omega)\}}{\sigma(\omega-\iota\omega)}e^{-\eta(\alpha+\beta)+\iota\eta(\beta-\alpha)}.\end{aligned}$$

Thus $\sigma_2(\alpha-\iota\beta)$ and $\sigma_2(\alpha+\iota\beta)$ are conjugate imaginaries.

Again, the function $\sigma_3(\alpha+\iota\beta)=\dfrac{\sigma\{\alpha+\iota(\beta+\omega)\}}{\sigma(\iota\omega)}e^{\iota\eta(\alpha+\iota\beta)}$ and the exponential factor is $e^{-\eta(\beta-\iota\alpha)}$. Also

$$\begin{aligned}\sigma_3(\alpha-\iota\beta)&=\frac{\sigma\{\alpha-\iota(\beta+\omega)+2\omega'\}}{\sigma(-\iota\omega+2\omega')}e^{\eta(\beta+\iota\alpha)}\\&=\frac{\sigma\{\alpha-\iota(\beta+\omega)\}}{\sigma(-\iota\omega)}e^{2\eta'(\alpha-\iota\beta)+\eta(\beta+\iota\alpha)}\\&=\frac{\sigma\{\alpha-\iota(\beta+\omega)\}}{\sigma(-\iota\omega)}e^{-\eta(\beta+\iota\alpha)}.\end{aligned}$$

Thus $\sigma_3(\alpha+\iota\beta)$ and $\sigma_3(\alpha-\iota\beta)$ are conjugate imaginaries, and hence it follows that z_0, z_0' being conjugate imaginaries, Z_0, Z_0' are so, and inversely.

16. The fluid motion is given by

$$\Psi+\iota\Phi=\frac{m}{2\pi}\log\frac{Z-Z_0}{Z-Z_0'}=\frac{m}{2\pi}\log\left(\frac{\sigma_2^2z}{\sigma_1z\sigma_3z}-\frac{\sigma_2^2z_0}{\sigma_1z_0\sigma_3z_0}\right)\Big/\left(\frac{\sigma_2^2z}{\sigma_1z\sigma_3z}-\frac{\sigma_2^2z_0'}{\sigma_1z_0'\sigma_3z_0'}\right). \tag{33}$$

Remarking that $\qquad \wp z-1=\sigma_1^2z/\sigma^2z,$

and $\qquad \wp z+1=\sigma_3^2z/\sigma^2z,$

so that $\qquad \sigma_1^2z+\sigma_3^2z=2\sigma^2z\wp z=2\sigma^2z,$

equation (33) becomes

$$\Psi+\iota\Phi=\frac{m}{2\pi}\log\frac{(\sigma_1z\sigma_1z_0-\sigma_3z\sigma_3z_0)(\sigma_1z\sigma_3z_0-\sigma_3z\sigma_1z_0)}{(\sigma_1z\sigma_1z_0'-\sigma_3z\sigma_3z_0')(\sigma_1z\sigma_3z_0'-\sigma_3z\sigma_1z_0')}. \tag{34}$$

The factor $\sigma_1z\sigma_3z_0-\sigma_1z_0\sigma_3z$ of the numerator may be put into factors. Using

Schwarz's formula [D][9], p. 81, and taking first $\mu = 1$, $\lambda = 3$, and then $\mu = 3$, $\lambda = 1$, we have

$$\sigma_1 z \sigma_3 z_0 = \sigma_3 \frac{z+z_0}{2} \sigma_1 \frac{z+z_0}{2} \sigma_3 \frac{z-z_0}{2} \sigma_1 \frac{z-z_0}{2}$$

$$- (e_1 - e_3)\, \sigma \frac{z+z_0}{2} \sigma_2 \frac{z+z_0}{2} \sigma \frac{z-z_0}{2} \sigma_2 \frac{z-z_0}{2},$$

$$\sigma_1 z_0 \sigma_3 z = \sigma_1 \frac{z+z_0}{2} \sigma_3 \frac{z+z_0}{2} \sigma_1 \frac{z-z_0}{2} \sigma_3 \frac{z-z_0}{2}$$

$$- (e_3 - e_1)\, \sigma \frac{z+z_0}{2} \sigma_2 \frac{z+z_0}{2} \sigma \frac{z-z_0}{2} \sigma_2 \frac{z-z_0}{2},$$

whence

$$\sigma_1 z \sigma_3 z_0 - \sigma_1 z_0 \sigma_3 z = 2(e_1 - e_3)\, \sigma \frac{z+z_0}{2} \sigma \frac{z-z_0}{2} \sigma_2 \frac{z+z_0}{2} \sigma_2 \frac{z-z_0}{2}.$$

So that the fluid motion is given by

$$\Psi + \iota\Phi = \frac{m}{2\pi} \log \frac{(\sigma_1 z \sigma_1 z_0 - \sigma_3 z \sigma_3 z_0)\, \sigma \frac{1}{2}(z+z_0)\, \sigma \frac{1}{2}(z-z_0)\, \sigma_2 \frac{1}{2}(z+z_0)\, \sigma_2 \frac{1}{2}(z-z_0)}{(\sigma_1 z \sigma_1 z_0' - \sigma_3 z \sigma_3 z')\, \sigma \frac{1}{2}(z+z')\, \sigma \frac{1}{2}(z-z_0')\, \sigma_2 \frac{1}{2}(z+z_0')\, \sigma_2 \frac{1}{2}(z-z_0')}, \tag{35}$$

17. It may be noticed that in the notation of Jacobi's elliptic functions

$$Z = \frac{\mathrm{dn}^2 z}{\mathrm{cn}\, z}, \quad k = \frac{1}{\sqrt{2}}. \tag{35a}$$

So that

$$\Psi + \iota\Phi = \frac{m}{2\pi} \log \left\{ \left(\frac{\frac{1}{2}(1 + \mathrm{cn}^2 z)}{\mathrm{cn}\, z} - \frac{\frac{1}{2}(1 + \mathrm{cn}^2 z_0)}{\mathrm{cn}\, z_0} \right) \Big/ \left(\frac{\frac{1}{2}(1 + \mathrm{cn}^2 z)}{\mathrm{cn}\, z} - \frac{\frac{1}{2}(1 + \mathrm{cn}^2 z_0')}{\mathrm{cn}\, z_0'} \right) \right\},$$

or

$$\Psi + \iota\Phi = \frac{m}{2\pi} \log \frac{(\mathrm{cn}\, z - \mathrm{cn}\, z_0)(1 - \mathrm{cn}\, z\, \mathrm{cn}\, z_0)}{(\mathrm{cn}\, z - \mathrm{cn}\, z_0')(1 - \mathrm{cn}\, z\, \mathrm{cn}\, z_0')}, \tag{36}$$

is the solution for a vortex at z_0.

As an example of the application of the method when the motion is due to sources or sinks, the solution for a source and sink each of strength m at the base angles marked 1 and -1 is

$$\Phi + \iota\Psi = \frac{m}{\pi} \log \frac{1 + \mathrm{cn}\, z}{1 - \mathrm{cn}\, z}. \tag{37}$$